# Quantum Radiation from Black Holes and Naked Singularities in Spherical Dust Collapse

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## Abstract

A sufficiently massive collapsing star will end its life as a spacetime singularity. The nature of the Hawking radiation emitted during collapse depends critically on whether the star's boundary conditions are such as would lead to the eventual formation of a black hole or, alternatively, to the formation of a naked singularity. This latter possibility is not excluded by the singularity theorems. We discuss the nature of the Hawking radiation emitted in each case. We justify the use of Bogoliubov transforms in the presence of a Cauchy horizon and show that if spacetime is assumed to terminate at the Cauchy horizon, the resulting spectrum is thermal, but with a temperature different from the Hawking temperature.

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## 1. Introduction.

There are by now many known examples of formation of naked singularities in spherical gravitational collapse in classical general relativity.[1,2] Whereas these examples do not necessarily invalidate the Cosmic Censorship Hypothesis,[3] it is interesting to ask what a star forming a naked singularity would look like to a distant observer. Since visible regions of very high curvature develop during the collapse, it can be expected that quantum effects will play a significant role in determining the evolution of the star. Furthermore, it should be possible to describe these quantum effects using techniques from quantum field theory up to the time when curvatures approach Planck scales.

A principal issue is a comparison between the Hawking evaporation of a star that forms a black hole, and the corresponding quantum evaporation of a star that forms a naked singularity. Studies of this problem can be divided into two classes; one in which the vacuum expectation value (VEV) of the energy momentum tensor of a quantized field is calculated in the background of the classical collapsing star using the trace anomaly, and the other in which the VEV of the radiation flux and spectrum of the radiation is calculated asymptotically in the geometric optics approximation, using Bogoliubov coefficients.[4]

A few studies have been carried out in recent years to calculate the VEV of the quantized stress tensor in spacetimes which evolve to naked singularities. The central idea here is to investigate the behavior of the VEV in the approach to the Cauchy horizon. It has been found in all examples of shell focussing naked singularities that the flux of radiation diverges as the Cauchy horizon is approached. The divergence of the outgoing flux of the quantum field on the Cauchy horizon would suggest that the back-reaction ultimately prevents the naked singularity from forming. Furthermore, a divergent flux could in principle be measured by a distant observer if such objects were to occur in nature.

Perhaps the first investigation in this context was due to Ford and Parker,[5] who calculated the outgoing flux of a quantized massless scalar field in the spacetime of a collapsing spherical dust cloud which develops a shell-crossing naked singularity. The flux was calculated using the geometric optics approximation, and remains finite in the approach to the formation of the naked singularity. Another early calculation was by Hiscock et al.,[6] who computed the outgoing flux for a massless scalar field in the two dimensional self-similar Vaidya spacetime (obtained by suppressing angular coordinates in the spherical spacetime) which evolves to a shell-focusing naked singularity. In this instance, the flux diverges on the Cauchy horizon. In recent works[7] we calculated the outgoing flux in the background spacetime of a spherical self-similar dust cloud with a naked singularity, using both the geometric optics approximation (analogous to Ford and Parker[5]) and the trace anomaly (analogous to Hiscock et al.[6]). Again, the flux diverges on the Cauchy horizon.

As mentioned above, apart from the calculation of the quantum stress tensor, a calculation of the spectrum of the created particles is also of great interest. Since the calculation of the stress tensor is local, it can be carried out using standard techniques of quantum theory. However, the presence of the Cauchy horizon raises subtle issues in the calculation of the spectrum, and the purpose of the present paper is to address some of these issues.

Consider the Penrose diagram for a collapsing star which develops a naked singularity (figure 1).

Figure 1. Formation of a naked singularity

The presence of a Cauchy horizon implies that the region of $\mathcal{I}^+$ to the future of the Cauchy horizon is exposed to the naked singularity. Hence one might conclude that a complete set of modes cannot be defined on $\mathcal{I}^+$, and the standard Bogoliubov calculation cannot be carried out[6].

There are two ways out of this apparent obstacle. One is to consider the very real possibility that a star collapsing to a naked singularity does not destroy the whole universe. $\mathcal{I}^+$ continues to be well-defined beyond the Cauchy horizon, and since no ingoing modes reach out to this part of $\mathcal{I}^+$, the outgoing modes in this region can be set to zero. It has been shown by us that the resulting spectrum is non-thermal.[8] In section 2 we give a justification for this approach by showing that the total radiation computed from this spectrum indeed equals the integrated flux obtained from the stress tensor.

A second way out of the apparent obstacle is to assume that the Cauchy horizon is actually the end of spacetime, *i.e.,* that the analytical continuation which defines the space time to its future is not physical. We show in section 4 that in this case the spectrum is black-body, with a temperature different from that for the Hawking black hole. This case is analogous to that of a marginally naked singularity (treated in ref. [6]), *i.e.,* one in which the Cauchy horizon coincides with the event horizon.

Another issue in the calculation of the spectrum concerns the existence of a complete orthonormal basis set of infalling waves on the Cauchy horizon. In

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principle, the quantum field should be expressible in complete bases on $\mathcal{I}^+ \cup \mathcal{H}_C$, where $\mathcal{H}_C$ is the Cauchy horizon. There is an obvious difficulty in constructing a basis of infalling waves on $\mathcal{H}_C$ on account of the central singularity, and so there is an essential ambiguity in evolving the field in the future of the Cauchy horizon. This ambiguity is inconsequential, for it would be relevant if one were interested in constructing outgoing wave packets on $\mathcal{I}^+$ in the future of the Cauchy horizon, from infalling waves on $\mathcal{I}^-$. No such outgoing packets exist, however, because all probes coming in from $\mathcal{I}^-$ at such late advanced times would be absorbed by the singularity. We are therefore only interested in complete orthonormal sets of solutions to the wave equation on hypersurfaces in the past of the Cauchy horizon. These are well defined. Moreover, the spectrum on $\mathcal{I}^+$ will be independent of any particular choice of infalling basis states.

We will compare the features of the Hawking radiation from black holes and naked singularities via a model of self-similar collapse based on inhomogeneous dust which is described in section 3. Our reason for using this particular model is twofold: firstly, the causal structure of the spacetime is well understood[9] and secondly, the model exhibits both classical black hole and naked singularity end states, thus allowing for a comparison between the behaviors of each case within a unified picture. We do not expect any of the conclusions to be heavily dependent on the model itself, based as they are on general arguments that would apply whenever black holes or naked singularities are formed.

## 2. Relationship between the flux and spectrum.

Consider the propagation of a scalar field in some spherically symmetric, asymptotically flat background spacetime. For convenience we consider a massless scalar field, though massive fields as well as fields of arbitrary spin may be treated using the same techniques. We follow closely the treatment of Ford and Parker[5] (see

also Birrell and Davies in ref. [4]). In this spacetime, we will assume that the radial null rays define a one-to-one mapping between a portion of past null infinity, $\mathcal{I}^-$, and some region of future null infinity, $\mathcal{I}^+$. We will also assume that a complete basis set of null infalling rays may be defined on $\mathcal{I}^-$ and a complete basis of null outgoing rays defined on $\mathcal{I}^+$. Let $t, r, \theta$ and $\phi$ define a quasi-Minkowskian coordinate system in the asymptotic region and let $\tilde{U} = t - r$ and $\tilde{V} = t + r$ be the the null coordinates there. We imagine that a null incoming ray, at $\tilde{V} = \text{constant}$, originates on $\mathcal{I}^-$ and propagates through the spacetime geometry, turning into a null outgoing ray, $\tilde{U} = \text{constant}$, on $\mathcal{I}^+$ with value $\tilde{U} = F(\tilde{V})$. In the time reversed situation, one could trace a null ray, $\tilde{U} = \text{constant}$, on $\mathcal{I}^+$ into the past. Such a ray would have originated at $\tilde{V} = G(\tilde{U})$ on $\mathcal{I}^-$, so that $G(\tilde{U})$ is the inverse of $F(\tilde{V})$. In Minkowski space, for example, $F(\tilde{V}) = \tilde{V}$ and $G(\tilde{U}) = \tilde{U}$. If the functions, $F(\tilde{V})$, are known or can be determined, one considers positive energy solutions of the massless wave equation which have the following asymptotic form

$$\overline{f}_{\omega l m} \sim \frac{1}{\sqrt{4\pi\omega}r} \left[ e^{-i\omega\tilde{U}} + e^{-i\omega F(\tilde{V})} \right] Y_{lm}(\theta, \phi), \tag{2.1}$$

which corresponds to an outgoing plane wave on $\mathcal{I}^+$ and is normalized on a spatial hypersurface in the asymptotically flat "out" region according to

$$\langle \overline{f}_{\omega l m}, \overline{f}_{\omega' l' m'} \rangle = \delta(\omega - \omega')\delta_{ll'}\delta_{mm'}, \tag{2.2}$$

with the inner product being defined by

$$\langle \overline{f}, \overline{h} \rangle = -i \int_{\Sigma} d\Sigma^\mu \sqrt{g_\Sigma} \left[ \overline{f}(\partial_\mu \overline{h}^*) - (\partial_\mu \overline{f})\overline{h}^* \right], \tag{2.3}$$

$\Sigma$ being the hypersurface. Wavepackets formed from the $\overline{f}_{\omega l m}$ are outgoing plane waves at late times and incoming at early times in accordance with popagation by

geometrical optics, and any positive energy solution of the scalar wave equation that is outgoing on $\mathcal{I}^+$ can be written as a wavepacket formed from the $\overline{f}_{\omega lm}$. An equivalent expression can be given for wavepackets that are incoming plane waves on $\mathcal{I}^-$. These will be formed from

$$f_{\omega lm} \;\sim\; \frac{1}{\sqrt{4\pi\omega r}} \left[ e^{-i\omega\tilde{V}} \;+\; e^{-i\omega G(\tilde{U})} \right] Y_{lm}(\theta,\phi). \tag{2.4}$$

A quantum field, $\phi$, may therefore be expanded in either basis

$$\begin{aligned}
\phi \;&=\; \sum_{lm} \int_0^\infty d\omega [\overline{a}_{\omega lm}\overline{f}_{\omega lm} \;+\; \overline{a}_{\omega lm}^\dagger \overline{f}_{\omega lm}^*] \\
\;&=\; \sum_{lm} \int_0^\infty d\omega [a_{\omega lm} f_{\omega lm} \;+\; a_{\omega lm}^\dagger f_{\omega lm}^*],
\end{aligned} \tag{2.5}$$

in terms of the annihilation and creation operators, $a_{\omega lm}$ and $\overline{a}_{\omega lm}$, and their hermitean conjugates. The vacuum defined by the $a_{\omega lm}$, according to $a_{\omega lm}|0\rangle = 0$, is the "in" vacuum and that defined by the $\overline{a}_{\omega lm}$ is the "out" vacuum. One is normally interested in the production of particles on $\mathcal{I}^+$, $i.e.$, in the quantity $\langle 0|\overline{N}_{\omega lm}|0\rangle$ as $\tilde{V} \to \infty$, where $\overline{N}_{\omega lm} = \overline{a}_{\omega lm}^\dagger \overline{a}_{\omega lm}$ is the number operator in the "out" vacuum. It is then easily shown that this quantity is determined by the second of the two Bogoliubov coefficients,

$$\begin{aligned}
\alpha_{\omega\omega'} \;&=\; \langle f_w, \overline{f}_{w'} \rangle \\
\beta_{\omega\omega'} \;&=\; -\langle f_\omega, \overline{f}_{\omega'}^* \rangle,
\end{aligned} \tag{2.6}$$

that relate the two descriptions of the quantum field in (2.5), according to

$$\langle 0|\overline{N}_\omega|0\rangle \;=\; \int_0^\infty d\omega' |\beta(\omega'\omega)|^2. \tag{2.7}$$

We have suppressed the dependence on $l, m$ because the geometric optics approximation that will be used in this paper is invalid for higher angular momentum

modes. However, these modes are expected to contribute little either to the flux or to the spectrum of the radiation because of the centrifugal potential, which would cause them to scatter to infinity before they encounter the region of high curvature(see, for example, B. S. DeWitt in ref.[4]) In the "out" region,

$$
\begin{aligned}
f_\omega &\approx \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega G(\tilde{U})} \\
\overline{f}_\omega &\approx \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega \tilde{U}},
\end{aligned}
\tag{2.8}
$$

giving

$$
\beta(\omega'\omega) = \frac{1}{2\pi}\sqrt{\frac{\omega}{\omega'}} \int_{-\infty}^{\tilde{U}_o} d\tilde{U} e^{-i\omega\tilde{U}} e^{-i\omega' G(\tilde{U})},
\tag{2.9}
$$

where we have used (2.6), and where $\tilde{U} = \tilde{U}_o$ represents the last outgoing ray that originated in an incoming packet from $\mathcal{I}^-$. An equivalent and alternative expression, which constructs $\beta(\omega'\omega)$ on $\mathcal{I}^-$, is

$$
\beta(\omega'\omega) = \frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{\tilde{V}_o} d\tilde{V} e^{-i\omega'\tilde{V}} e^{-i\omega F(\tilde{V})},
\tag{2.10}
$$

using the asymptotic forms

$$
\begin{aligned}
f_\omega &\approx \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega\tilde{V}} \\
\overline{f}_\omega &\approx \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega F(\tilde{V})}
\end{aligned}
\tag{2.11}
$$

on $\mathcal{I}^-$, and where $\tilde{V} = \tilde{V}_o$ is the last incoming ray that turns into an outgoing packet on $\mathcal{I}^+$.

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One may now compute the components of the stress energy tensor of the scalar field from the usual expression

$$\langle 0|T_{\mu\nu}(x)|0\rangle = \lim_{x' \to x} \mathcal{D}_{\mu\nu'}\frac{1}{2}G^{(1)}(x, x'), \tag{2.12}$$

where $G^{(1)}(x, x')$ is Hadamard's elementary function,

$$G^{(1)}(x, x') = \langle 0|\{\phi(x), \phi(x')\}|0\rangle, \tag{2.13}$$

and $\mathcal{D}_{\mu\nu'}(x, x')$ is a non-local operator defined by the form of the stress-energy tensor and point-splitting. The expression in (2.12) must obviously be regularized and then renormalized, upon which one obtains, in particular, an expression for the radiated flux on $\mathcal{I}^{+}$,[4]

$$T_{\tilde{U}\tilde{U}}(\tilde{V}) = \frac{1}{24\pi}\left[\frac{F'''}{F'^3} - \frac{3}{2}\left(\frac{F''}{F'^2}\right)^2\right], \tag{2.14}$$

and a corresponding relation in terms of $G(\tilde{U})$.

It is more interesting, however, to recover the radiated flux, (2.14), from a consistency condition: the integrated flux over $\mathcal{I}^{+}$ must equal the total radiated energy as calculated by integrating the radiation spectrum over all frequencies, $i.e.$,

$$\int_{-\infty}^{\tilde{U}_o} d\tilde{U}\langle 0|T_{\tilde{U}\tilde{U}}|0\rangle = \int_0^\infty d\omega\,\omega\langle 0|\overline{N}(\omega)|0\rangle = \int_0^\infty d\omega\,\omega\int_0^\infty d\omega'\,\beta^*(\omega'\omega)\beta(\omega'\omega). \tag{2.15}$$

Using $\tilde{U} = F(\tilde{V})$, we can transform the integral over future null infinity to one over past null infinity, and write

$$\int_{-\infty}^{\tilde{V}_o} d\tilde{V}\,F'(\tilde{V})\langle 0|T_{\tilde{U}\tilde{U}}|0\rangle = \int_0^\infty d\omega\,\omega\langle 0|\overline{N}(\omega)|0\rangle$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\tilde{V}_o} d\tilde{V}\int_{-\infty}^{\tilde{V}_o} d\tilde{V}'\int_0^\infty d\omega\int_0^\infty d\omega'\,\omega'\,e^{i\omega'(\tilde{V}'-\tilde{V})}e^{i\omega[F(\tilde{V}')-F(\tilde{V})]}. \tag{2.16}$$

While it may seem rather indirect, our reason for computing the radiated flux in this way is the following. When the singularity is globally naked, the Cauchy horizon will intersect future null infinity in the retarded past of the apparent horizon, at some point, say $\tilde{U}_o$. This means that the integration in expression (2.9) for $\beta(\omega'\omega)$ will not extend over all of $\mathcal{I}^+$, as it does for the black hole, but only the portion of $\mathcal{I}^+$ that is in the retarded past of $\tilde{U} = \tilde{U}_o$. It is because, as we have pointed out in the introduction, any ray originating at such a value of $\tilde{V}$ on $\mathcal{I}^-$ as would translate into an outgoing ray in the future of $\tilde{U} = \tilde{U}_o$ would never arrive at $\mathcal{I}^+$, being, instead, absorbed by the singularity. If (2.16), for any $\tilde{V}_o$, reproduces the correct expression, (2.14), for the radiated flux as computed by a direct application of (2.12), it increases our confidence in the spectrum obtained from (2.7) even when the singularity is globally naked. $\tilde{V}_o$ represents the last null ray that originates on $\mathcal{I}^-$ and is able to reach $\mathcal{I}^+$. Both sides of equation (2.15) are infinite, but this does not trouble us as we are interested only in obtaining the flux. The infinite result is because, when the back reaction of spacetime is not accounted for, particle production will occur indefinitely even though the system loses energy. Energy conservation requires, therefore, that the back reaction will dominate at some stage. Before this stage is reached, however, (2.16) should serve as a good approximation of the actual physical situation.

The integrals over $\omega$ and $\omega'$, on the right hand side of (2.16), can be performed to yield,

$$\int_{-\infty}^{\tilde{V}_o} d\tilde{V} F'(\tilde{V}) \langle 0|T_{\tilde{U}\tilde{U}}|0\rangle = \frac{-i}{4\pi^2} \int_{-\infty}^{\tilde{V}_o} d\tilde{V} \int_{-\infty}^{\tilde{V}_o} d\tilde{V}' \frac{1}{(\tilde{V}' - \tilde{V})^2 [F(\tilde{V}') - F(\tilde{V})]}. \quad (2.17)$$

The integral on the right has a pole at $\tilde{V}' = \tilde{V}$ and contributions from points $\tilde{V}' \neq \tilde{V}$ vanish identically (as is seen by interchaging $\tilde{V}$ and $\tilde{V}'$ in (2.17)). Calling

$z = \tilde{V} - \tilde{V}'$, the r.h.s. of (2.17) becomes

$$\frac{-i}{4\pi^2} \int\limits_{-\infty}^{\tilde{V}_o} d\tilde{V} \int\limits_{\tilde{V}-\tilde{V}_o}^{\infty} dz \frac{1}{z^2[F(\tilde{V}-z) - F(\tilde{V})]}. \qquad (2.18)$$

We will define the $z$ integral as the contribution from the portion of the contour, in the complex $z$−plane, shown in figure 2, that runs from $\tilde{V} - \tilde{V}_o$ to $-\epsilon$ along the real line (II), along the infinitesimal semi-circle, $C_o$, of radius $\epsilon$ around the origin in the upper half plane, and from $+\epsilon$ to infinity along the real line (III). The non-vanishing contribution from the $z$−integral in (2.18) is then just $-\pi i b_{-1}(\tilde{V})$ where where $b_{-1}(\tilde{V})$ is the residue of the integrand at $z = 0$.
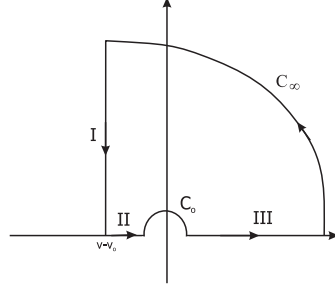


Figure 2. The Contour $C$

This is easily seen as follows: the value of the contour integral is identically zero, i.e., $I + II + III + C_o = 0$, as is the contribution from the semi-circle at infinity, $C_\infty$. Thus we may write the integral in (2.18) as

$$\int\limits_{-\infty}^{\tilde{V}_o} d\tilde{V} \int\limits_{\tilde{V}-\tilde{V}_o}^{\infty} dz \frac{1}{z^2[F(\tilde{V}-z) - F(\tilde{V})]} \;=\; -\int\limits_{-\infty}^{\tilde{V}_o} d\tilde{V} \int\limits_{I} dz \frac{1}{z^2[F(\tilde{V}-z) - F(\tilde{V})]}. \qquad (2.19)$$
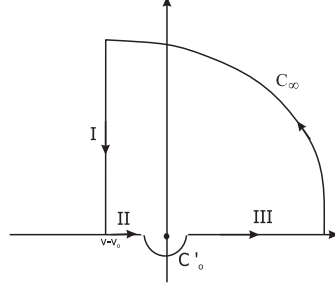
Next, consider the contour shown in figure 3.

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Figure 3. The Contour $C'$

The value of this contour integral is $2\pi i b_{-1}(\tilde{V})$, i.e., $I+II+III+C'_o = 2\pi i b_{-1}(\tilde{V})$. Combining these two results, one finds $I = \pi i b_{-1}(\tilde{V})-(II+III)$. However, because

$$\int\limits_{-\infty}^{\tilde{V}_o} d\tilde{V} \int\limits_{II+III} dz \frac{1}{z^2[F(\tilde{V}-z)-F(\tilde{V})]} \quad = \quad 0, \tag{2.20}$$

it follows immediately that value of the integral, as defined above, is

$$\frac{-i}{4\pi^2} \int\limits_{-\infty}^{\tilde{V}_o} d\tilde{V} \int\limits_{\tilde{V}-\tilde{V}_o}^{\infty} dz \frac{1}{z^2[F(\tilde{V}-z)-F(\tilde{V})]} \quad = \quad -\frac{1}{4\pi} \int\limits_{-\infty}^{\tilde{V}_o} d\tilde{V} b_{-1}(\tilde{V}). \tag{2.21}$$

The residue can be evaluated by expanding the integrand about $z = 0$. We have

$$\frac{1}{z^2[F(\tilde{V}-z)-F(\tilde{V})]} \quad = \quad \frac{1}{z^3}\left[1+\frac{z}{2!}\frac{F''}{F'}-\frac{z^2}{3!}\frac{F'''}{F'}+\frac{z^2}{(2!)^2}\left(\frac{F''}{F'}\right)^2+ \ ... \right], \tag{2.22}$$

giving

$$b_{-1}(\tilde{V}) \quad = \quad \frac{1}{6F'}\left[\frac{3}{2}\left(\frac{F''}{F'}\right)^2-\frac{F'''}{F'}\right], \tag{2.23}$$

or, inserting (2.23) into (2.21) and using (2.17), precisely the result in (2.14). The flux was originally obtained from (2.12) using standard point-splitting techniques. We have recovered it directly from a consistency condition, independently of details of the collapse.

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## 3. The Collapse of Inhomogeneous Dust.

We will apply the expressions obtained in (2.9) and (2.23) for the radiation spectrum and flux respectively, to the marginally bound, self-similar collapse of inhomogeneous dust. Although this model has been examined in detail by us elsewhere,[6,7] we include here a brief analysis of the causal structure of the space-time, both for the sake of completeness as well as to set our notation for the succeeding sections. It is described by the stress energy tensor

$$T_{\mu\nu} \quad = \quad \epsilon(t,r)\delta^0_\mu\delta^0_\nu. \tag{3.1}$$

The metric is well known and given in comoving coordinates[10] by

$$ds^2 \quad = \quad dt^2 \; - \; \tilde{R}'^2(t,r)dr^2 \; - \; \tilde{R}^2(t,r)d\Omega^2, \tag{3.2}$$

where the dust cloud is thought of as made up of concentric shells, each labeled by $r$. $\tilde{R}'(t,r)$ is the derivative of $\tilde{R}(t,r)$ with respect to $r$ and $\tilde{R}(t,r)$ is the physical radius (the area of a shell labelled by $r$ is $4\pi R^2(t,r)$) obeying, in the particular case of the marginally bound self similar collapse,

$$\tilde{R}(t,r) \quad = \quad r\left[1 \; - \; \frac{3\sqrt{\lambda}}{2}\frac{t}{r}\right]^{2/3}. \tag{3.3}$$

The physical radius is seen to depend on one parameter, $\lambda$, (the "mass parameter"). This parameter determines the total mass, $M(r)$, lying within the shell labeled by $r$ as $2GM(r) = \lambda r$. The total mass of the dust is therefore $2GM = \kappa = \lambda r_o$ where $r_o$ labels the outer boundary of the cloud. Now it can be shown that $\tilde{R}(t,r) = 0$ is a curvature singularity. This means that the singularity curves are $t_o(r) = 2r/(3\sqrt{\lambda})$, so that the last shell becomes singular at the time $t_o = 2/3\sqrt{r_o^3/\kappa}$.

Beyond $r = r_o$ spacetime is described by the the Schwarzschild solution

$$ds^2 = \left(1 - \frac{\kappa}{R}\right)dT^2 - \left(1 - \frac{\kappa}{R}\right)^{-1}dR^2 - R^2 d\Omega^2 \tag{3.4}$$

and the first and second fundamental forms of the two patches must be matched at the boundary. This has been done in the past and gives

$$
\begin{aligned}
T_o(t) &= -2\sqrt{\kappa \tilde{R}_o} - \frac{2}{3}\tilde{R}_o\sqrt{\frac{\tilde{R}_o}{\kappa}} + \kappa\ln\left|\frac{\sqrt{\tilde{R}_o} + \sqrt{\kappa}}{\sqrt{\tilde{R}_o} - \sqrt{\kappa}}\right| \\
&= t - \frac{2}{3\sqrt{\kappa}}r_o^{3/2} - 2\sqrt{\kappa\tilde{R}_o} + \kappa\ln\left|\frac{\sqrt{\tilde{R}_o} + \sqrt{\kappa}}{\sqrt{\tilde{R}_o} - \sqrt{\kappa}}\right| \\
R_o(t) &= r_o\left[1 - a\frac{t}{r_o}\right]^{2/3},
\end{aligned}
\tag{3.5}
$$

where we have set $a = 3\sqrt{\lambda}/2$.

For the marginally bound, self-similar collapse under consideration, it is relatively simple to find null coordinates for this system. Consider the effective two dimensional metric,

$$ds^2 = dt^2 - \tilde{R}'^2(t,r)dr^2, \tag{3.6}$$

and change variables to $z, x$ where $z = \ln r$, $x = t/r$. This gives

$$
\begin{aligned}
ds^2 &= r^2\left[dx^2 + 2xdxdz + (x^2 - \tilde{R}'^2(x))dz^2\right] \\
&= r^2(x^2 - \tilde{R}'^2)(d\tau^2 - d\chi^2),
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
\tau &= z + \frac{1}{2}(I_- + I_+) \\
\chi &= \frac{1}{2}(I_- - I_+),
\end{aligned}
\tag{3.8}
$$

in terms of

$$I_\pm(x) = \int\frac{dx}{x \pm \tilde{R}'}. \tag{3.9}$$

We would like to choose null coordinates such that in the limit as $\lambda \to 0$ these

reduce to the standard null coordinates in Minkowski space. Such coordinates are given by

$$u = \begin{cases} +re^{I-} & x - \tilde{R}' > 0 \\ -re^{I-} & x - \tilde{R}' < 0 \end{cases}$$

$$v = \begin{cases} +re^{I+} & x + \tilde{R}' > 0 \\ -re^{I+} & x + \tilde{R}' < 0 \end{cases} \tag{3.10}$$

To further analyze the causal structure, it is now convenient to go over to the variable $y$ defined by $y = \sqrt{\tilde{R}/r}$. In terms of $y$, the integrals $I_\pm$ can be written as

$$I_\pm = 9 \int \frac{y^3 dy}{3y^4 \mp ay^3 - 3y \mp 2a} \tag{3.11}$$

and the coordinates (3.10) become

$$u = \begin{cases} +re^{I-} & f_-(y) < 0 \\ -re^{I-} & f_-(y) > 0 \end{cases}$$

$$v = \begin{cases} +re^{I+} & f_+(y) < 0 \\ -re^{I+} & f_+(y) > 0 \end{cases} \tag{3.12}$$

where

$$f_\pm(y) = 3y^4 \mp ay^3 - 3y \mp 2a. \tag{3.13}$$

Let $\alpha_i^\pm$ be the roots of $f_\pm(y)$, for $i \in \{1, 2, 3, 4\}$. As $f_\pm$ are both real, they admit either 0, 2, or 4 real roots. The integrals can now be put in the form

$$I_\pm = 3 \int dy \left[ \sum_{i=1}^{4} \frac{A_i^\pm}{(y - \alpha_i^\pm)} \right], \tag{3.14}$$

where the $A_i^\pm$ are constants related to the coefficients of $f_\pm(y)$ and their roots by,

$$A_i^\pm = \frac{\alpha_i^{\pm 3}}{f'_\pm(\alpha_i^\pm)}. \tag{3.15}$$

In particular, the $A_i^\pm$ satisfy $\sum_i A_i^\pm = 1$. If all the roots are real, the solution is

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explicitly given by

$$
\begin{aligned}
u(y) &= \pm r \prod_{i=1}^{4} |y - \alpha_i^-|^{3A_i^-} \\
v(y) &= \pm r \prod_{i=1}^{4} |y - \alpha_i^+|^{3A_i^+}.
\end{aligned}
\tag{3.16}
$$

We will now consider the case in which there are two real roots and a conjugate pair of complex roots. As we will shortly show at least two real roots (possibly degenerate) are required for the existence of a globally naked singularity at the origin so we do not consider the case when all the roots are complex even though it may be carried out in the same spirit. Let us order the roots so that the first two, $\alpha_{1,2}$, are a complex conjugate pair and $\alpha_{3,4}$ are real. From (3.15) it follows that $A_{1,2}$ is also a complex pair whereas $A_{3,4}$ are real. Then the integrals are of the form

$$
\begin{aligned}
I &= 3 \int dy \left[ \sum_{i=1}^{4} \frac{A_i}{(y - \alpha_i)} \right] \\
&= 3 \left[ A \ln(y - \alpha) + A^* \ln(y - \alpha^*) + \sum_{i=3,4} A_i \ln |y - \alpha_i| \right],
\end{aligned}
\tag{3.17}
$$

where $\alpha, \alpha^*$ are the complex roots and $A, A^*$ are the (complex) coefficients. Putting

$$
A = |A|e^{i\phi}, \qquad y - \alpha = |y - \alpha|e^{i\xi},
\tag{3.18}
$$

so that the $u, v$ coordinates have the explicit (and formal) solution

$$
\begin{aligned}
u(y) &= \pm r |y - \alpha^-|^{6|A^-|\cos\phi^-} e^{-6|A^-|\xi^- \sin\phi^-} \Pi_{i=3,4} |y - \alpha_i^-|^{3A_i^-} \\
v(y) &= \pm r |y - \alpha^+|^{6|A^+|\cos\phi^+} e^{-6|A^+|\xi^+ \sin\phi^+} \Pi_{i=3,4} |y - \alpha_i^+|^{3A_i^+}.
\end{aligned}
\tag{3.19}
$$

Consider the center $(r = 0)$ at early times, $t < 0$. Then, because $y = (1 -$

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$at/r)^{1/3} \to \infty$, (3.19) gives (when all roots are real)

$$
\begin{aligned}
u &\to -r|y|^{3\sum_i A_i^-} &= -r(1 - a\frac{t}{r}) &\to at \\
v &\to -r|y|^{3\sum_i A_i^+} &= -r(1 - a\frac{t}{r}) &\to at.
\end{aligned}
\tag{3.20}
$$

This line is therefore given by $u = v$. When two of the roots are complex conjugates of each other, the line is still $u = v$ as we now show. Note that

$$
\xi = \tan^{-1}\left(\frac{\mathrm{Im}(-\alpha)}{\mathrm{Re}(y - \alpha)}\right)
$$

($y$ is real), so that as $y \to \infty$, $\xi \to 0$. Then clearly

$$
\begin{aligned}
u &\to -r|y|^{3(2\mathrm{Re}A^- + A_3^- + A_4^-)} \\
v &\to -r|y|^{3(2\mathrm{Re}A^+ + A_3^+ + A_4^+)}
\end{aligned}
\tag{3.21}
$$

but, since $\sum_i A_i^\pm = 1$, we have the same result as before.

The general solutions in (3.16) and (3.19) are useful to analyze another limit, namely the singularity at $r \to at$. This means that $y \to 0$. Now when $y \to 0$, $f_-(y) > 0$ and $f_+(y) < 0$. Then we see that (if all roots are real)

$$
\begin{aligned}
u &= -r\prod_i |\alpha_i^-|^{3A_i^-} \\
v &= r\prod_i |\alpha_i^+|^{3A_i^+}
\end{aligned}
\tag{3.22}
$$

and, in particular,

$$
\frac{v}{u} = -c = -\prod_i \frac{|\alpha_i^+|^{3A_i^+}}{|\alpha_i^-|^{3A_i^-}},
\tag{3.23}
$$

which is a negative constant, in general $\neq -1$. The singularity is therefore *spacelike* until the last shell, $r = r_o$, collapses at $t = t_o = r_o/a$. The case of a pair of conjugate complex roots trivially gives the same result. Beyond this point the

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singularity will be spacelike because it is just the Schwarzschild singularity in the exterior region. The behavior of the origin, $r = 0, t = 0$, is peculiar. It is the meeting point between two lines $u = v$ and $u = -cv$ and its nakedness (coveredness) is far from clear. However, if a null ray originating at this point reaches the boundary at Kruskal coordinate $U < 0$ in the Schwarzschild region, it will reach $\mathcal{I}^+$ and then the origin will be globally naked.

We will be interested in the earliest null ray leaving the singularity and reaching $\mathcal{I}^+$ (the Cauchy Horizon) as well as the earliest null ray that strikes the singularity from $\mathcal{I}^-$. These rays can be expected to intersect the first singular shell at $r = 0, t = 0$, so it is natural to carefully examine the null rays passing through this point. The origin, being the intersection of the lines $u = v$ and $v = -cu$ ($c \neq 1$ in general), corresponds to the point $u = 0 = v$. Now any null ray traveling toward $\mathcal{I}^+$ with $u = 0$ must have either $r = 0$ or $I_- \to -\infty$. Therefore, when $r \neq 0$, such a ray is possible if and only if $y = \alpha_k^-$ for some *real* root, $\alpha_k^-$ of the polynomial $f_-(y)$. Indeed such a root may not exist, in which case the singularity is not naked as no null rays can emanate from it. In this case, a black hole is formed, as shown in figure 3.
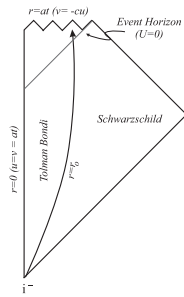


Figure 4. Formation of a black hole

If a real root exists however, at least one null ray leaves this point and reaches the boundary. The existence of real roots of the polynomials $f_-(y)$ is therefore a necessary condition for the nakedness of the origin. This places a constraint on the

possible value of the constant $a$ in the mass function. One finds that real roots exist provided that $a < a_c \sim 0.638$.[9] Each root corresponds to a null ray emanating from $u = 0 = v$ and there are at least two of them, if any at all. Because $y = \alpha_i$ implies that $t = r(1 - \alpha_i^3)/a$, we choose the largest real root of $f_-(y)$ as the one that gives the earliest null ray emanating from $u = 0 = v$ and call it $\alpha_-$. Thus, $y = \alpha_-$ is the Cauchy horizon.

A similar reasoning can now be given for the incoming rays passing through $u = 0 = v$. Again any ray with $v = 0$ for $r \neq 0$ must have $I_+ \to -\infty$, which is possible only if $y = \alpha_k^+$ for some *real* root, $\alpha_k^+$ of the polynomial $f_+(y)$. Now, $f_+(y)$ admits two real roots, one unphysical (negative) and one positive. Again, call the (positive) physical root $\alpha_+$.

What we have described above is pictured in the Penrose diagram of figure 1. It is to be expected that the behavior of quantum fields will be extremely sensitive to the collapse scenario being considered, that is to whether the mass parameter $a$ lies below or above its critical value, and this is the topic of the next section.

## 4. Radiation flux and spectrum for black holes and naked singularities.

We will henceforth consider rays in the neighborhood of the lines given by $y = \alpha_-$ for outgoing rays and $y = \alpha_+$ for incoming rays. The precise values of $\alpha_\pm$ in terms of the mass parameter will not interest us for this work but we will Taylor expand about these two values, considering $y_\pm = \tilde{y}_\pm + \alpha_\pm$.

Returning to (3.5), one can rewrite the Schwarzschild radial coordinate and time on the boundary as follows

$$
\begin{aligned}
R_o(y) &= r_o y^2 \\
T_o(y) &= -\frac{r_o}{a} y^3 - \frac{4}{3} a r_o y - \frac{4}{9} a^2 r_o \ln\left|\frac{3y/2a - 1}{3y/2a + 1}\right|.
\end{aligned}
\tag{4.1}
$$

Therefore, the Eddington-Finkelstein null coordinates on the boundary, $\tilde{U}_o(y) =$

$T_o(y) - R_{o*}(y)$, $\tilde{V}_o(y) = T_o(y) + R_{o*}(y)$, (where $R_{o*}$ is the tortoise coordinate) take the form

$$\begin{aligned}
\tilde{U}_o(y) &= -\frac{r_o}{a}y^3 - \frac{4}{3}ar_oy - r_oy^2 - \frac{8}{9}a^2r_o\ln|3y/2a - 1| \\
\tilde{V}_o(y) &= -\frac{r_o}{a}y^3 - \frac{4}{3}ar_oy + r_oy^2 + \frac{8}{9}a^2r_o\ln|3y/2a + 1|.
\end{aligned} \quad (4.2)$$

It is now clear that the earliest null outgoing ray, $u = 0$, from the origin (the Cauchy Horizon) within the cloud strikes the boundary at $y = \alpha_-$ and translates into the null outgoing ray

$$\tilde{U}_o^{(0)} = -\frac{r_o}{a}\alpha_-^3 - \frac{4}{3}ar_o\alpha_- - r_o\alpha_-^2 - \frac{8}{9}a^2r_o\ln|3\alpha_-/2a - 1|, \quad (4.3)$$

which is never infinite ($2a/3$ is not a root of $f_-(y)$). This null ray corresponds to a finite value of $\tilde{U}$ and will therefore reach $\mathcal{I}^+$, so the existence of real roots of $f_-(y)$ turns out to be not just necessary, but a sufficient condition for the origin to be globally naked. The same argument applies to the infalling ray(s): the earliest null ray to pass through the origin is the ray corresponding to the value $y = \alpha_+$, or

$$\tilde{V}_o^{(0)} = -\frac{r_o}{a}\alpha_+^3 - \frac{4}{3}ar_o\alpha_+ + r_o\alpha_+^2 + \frac{8}{9}a^2r_o\ln|3\alpha_+/2a + 1| \quad (4.4)$$

and, again, since $-2a/3$ is not a root of $f_+(y)$, $\tilde{V}$ is not infinitely negative and such a ray will have come from $\mathcal{I}^-$. Thus, the existence of a positive real root of $f_+(y)$ is sufficient to ensure that at least one infalling ray from $\mathcal{I}^-$ will intersect the origin.

The next question we must address is the relationship between the $\tilde{U}, \tilde{V}$ coordinates in the exterior and the $u, v$ coordinates (equations (18, 21)) on the boundary. This is difficult to do in general, but if we confine our study to rays that are "close" to $u = 0$ and $v = 0$ we can arrive at some conclusion regarding the quantum radiation on $\mathcal{I}^+$ near the Cauchy horizon. "Close" will be taken to mean linearizations about $y = \alpha_\pm$ respectively for incoming rays and outgoing rays.

First consider outgoing rays. For $y \sim \alpha_-$, define $y = \tilde{y} + \alpha_-$ and find that for small $\tilde{y}$

$$I_- \quad \sim \quad \gamma_- \ln \tilde{y} \quad + \quad \mathcal{O}(y), \tag{4.5}$$

where

$$\gamma_- \quad = \quad \frac{3\alpha_-^3}{f'_-(\alpha_-)}, \tag{4.6}$$

giving

$$u \quad = \quad -r|\tilde{y}|^{\gamma_-} \quad \rightarrow \quad y - \alpha_- \quad = \quad \left(-\frac{u}{r}\right)^{1/\gamma_-}. \tag{4.7}$$

Therefore in terms of $u$ (on the boundary) we can write $\tilde{U}$ as follows

$$\tilde{U} \quad \sim \quad \tilde{U}^{(0)}(\alpha_-) + \Gamma_-(\alpha_-)(y-\alpha_-) \quad = \quad \tilde{U}^{(0)}(\alpha_-) + \Gamma_-(\alpha_-)\left(-\frac{u}{r_o}\right)^{1/\gamma_-}, \tag{4.8}$$

where

$$\Gamma_- \quad = \quad -9\frac{r_o\alpha_-^3}{a(3\alpha_- - 2a)} \quad < \quad 0 \quad \text{when} \quad a < a_c. \tag{4.9}$$

Likewise, for incoming rays, put $y = \tilde{y} + \alpha_+$ and find that

$$I_+ \quad = \quad \gamma_+ \ln \tilde{y} \quad + \quad \mathcal{O}(y), \tag{4.10}$$

where

$$\gamma_+ \quad = \quad \frac{3\alpha_+^3}{f'_+(\alpha_+)}, \tag{4.11}$$

giving

$$v \quad = \quad -r|\tilde{y}|^{\gamma_+} \quad \rightarrow \quad y - \alpha_+ \quad = \quad \left(-\frac{v}{r}\right)^{1/\gamma_+}. \tag{4.12}$$

Thus, in terms of $v$ (on the boundary) we can write $\tilde{V}$ as follows

$$\tilde{V} \quad \sim \quad \tilde{V}^{(0)}(\alpha_+) + \Gamma_+(\alpha_+)(y-\alpha_+) \quad = \quad \tilde{V}^{(0)}(\alpha_+) + \Gamma_+(\alpha_+)\left(-\frac{v}{r_o}\right)^{1/\gamma_+}, \tag{4.13}$$

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where

$$\Gamma_+ \;=\; -\,9\frac{r_o\alpha_+^3}{a(3\alpha_+ + 2a)} \;\;<\;\; 0 \;\;\text{when}\;\; a < a_c. \tag{4.14}$$

We are now in a position to compute the radiated power close to the Cauchy horizon in the geometric optics approximation. consider a ray $\tilde{V} = \text{const.}$ in the infinite past. We are interested only in the region on $\mathcal{I}^+$ that is close to the Cauchy horizon, so the approximations in (4.8) and (4.13) will suffice. As the null ray crosses the boundary, we have

$$\tilde{V}(v) \;=\; \tilde{V}^{(0)} \;+\; \Gamma_+\left(-\frac{v}{r_o}\right)^{\frac{1}{\gamma_+}}. \tag{4.15}$$

This expression can be inverted to give

$$v(\tilde{V}) \;=\; -\,r_o\left[\frac{\tilde{V}^0 - \tilde{V}}{|\Gamma_+|}\right]^{\gamma_+}, \tag{4.16}$$

where we have used the fact that $\Gamma_+$ is negative. Next, reflecting about the center (here, $u = v$) gives

$$u(\tilde{V}) \;=\; -\,r_o\left[\frac{\tilde{V}^0 - \tilde{V}}{|\Gamma_+|}\right]^{\gamma_+}. \tag{4.17}$$

Now as the outgoing ray crosses the outer boundary, we have the relation

$$\tilde{U}(u) \;=\; \tilde{U}^{(0)} \;-\; \Gamma_-\left(-\frac{u}{r_o}\right)^{\frac{1}{\gamma_-}}$$

$$\rightarrow \;\; \tilde{U}(\tilde{V}) \;=\; \tilde{U}^{(0)} \;-\; |\Gamma_-|\left[\frac{\tilde{V}^0 - \tilde{V}}{|\Gamma_+|}\right]^{\frac{\gamma_+}{\gamma_-}}, \tag{4.18}$$

where now we have used the fact that $\Gamma_-$ is negative. Thus, the right hand side of

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(4.18) is $F(\tilde{V})$ and it has the form

$$F(\tilde{V}) \;=\; A \;-\; B(\tilde{V}^{(0)} - \tilde{V})^{\frac{\gamma_+}{\gamma_-}}, \tag{4.19}$$

where $B$ is a positive constant which is given in terms of the roots, $\alpha_\pm$ given before. We can now write down the power radiated as a function of $\tilde{V}$ using (2.14),

$$T_{\tilde{U}\tilde{U}}(\tilde{V}) \;\approx\; \frac{1}{48\pi B^2}\left[\frac{1-\gamma^2}{\gamma^2(\tilde{V}^{(0)} - \tilde{V})^{2\gamma}}\right] \qquad \gamma \neq 0, \tag{4.20}$$

where

$$\gamma \;=\; \frac{\gamma_+}{\gamma_-}. \tag{4.21}$$

The result can be written in terms of $\tilde{U}$ by inversion,

$$T_{\tilde{U}\tilde{U}}(\tilde{U}) \;\approx\; \frac{1}{48\pi}\left[\frac{1-\gamma^2}{\gamma^2(\tilde{U}^{(0)} - \tilde{U})^2}\right], \tag{4.22}$$

which diverges as the Cauchy horizon is approached.

Let us now consider the case when the origin is not naked, $i.e.$, all roots of the polynomial $f_-(y)$ are complex. This means of course that the Cauchy horizon is formed in the retarded future of the event horizon as shown in figure 4. ¿From the expression (4.3) for $\tilde{U}$, this implies that the event horizon intersects the boundary at $y \to 2a/3$. We will be interested in late times, so consider a ray that is close to the event horizon and that therefore intersects the boundary at $y = 2a/3 + \tilde{y}$. For such a ray,

$$\tilde{U} \;\sim\; -4M\ln|\tilde{y}|, \tag{4.23}$$

where $M = 2ar_o^2/9$ is the total mass of the cloud. Continuing backward into the cloud, it is necessary to retain only terms that are linear in $\tilde{y}$ in the expression for

$u$. This ray translates into the ray

$$u \; = \; r_o \exp(I_-(\frac{2a}{3} + \tilde{y})) \; = \; r_o \exp(\frac{3}{a}\tilde{y}) \; \sim \; r_o(1 + \frac{3\tilde{y}}{a}), \qquad (4.24)$$

or

$$\tilde{y} \; \sim \; \frac{a}{3}\left(\frac{u}{r_o} - 1\right), \qquad (4.25)$$

which, when substituted back into the expression for $\tilde{U}$ in (10), gives $\tilde{U}$ as a function of $u$ inside the cloud:

$$\tilde{U} \; = \; -4M\ln|\frac{a}{3}\left(\frac{u}{r_o} - 1\right)|. \qquad (4.26)$$

Reflecting at the center $(v = u)$, in terms of the advanced coordinate, $v$, within the cloud as

$$\tilde{U} \; = \; -4M\ln|\frac{a}{3}\left(\frac{v}{r_o} - 1\right)|. \qquad (4.27)$$

We must now find a relationship between $v$ and $\tilde{V}$ in the exterior region. Let us suppose that the ray $\tilde{U} = \infty$ traced backward to the ray $\tilde{V} = \tilde{V}_o$. It is not important to know the precise value of $\tilde{V}_o$ though this can be done and gives complicated expressions in terms of the roots of the polynomials $f_\pm(y)$. It is clear that this value, $\tilde{V}_o$ of $\tilde{V}$ corresponds to some given value $y_o$ of $y$ on the boundary. Consider a linearization about this value $y_o$ $(y = y_o + \tilde{y})$ so that

$$\tilde{V} \; = \; \tilde{V}_o \; + \; \tilde{V}'(y_o)\tilde{y} \; + \; ... \qquad (4.28)$$

where $\tilde{V}'(y)$ is the derivative of $\tilde{V}$ w.r.t. $y$. Again, expanding $v$ about this value, $y_o$ of $y$ on the boundary $r = r_o$ gives

$$v \; = \; r_o \exp(I_+(y_o + \tilde{y})) \; = \; v_o \; + \; v'(y_o)\tilde{y} \; + \; ... \qquad (4.29)$$

The precise values of $\tilde{V}_o, v_o, \tilde{V}'(y_o)$ and $v'(y_o)$ will not interest us for the following

analysis. What is important is that

$$v \;=\; v_o \;+\; \frac{v'(y_o)}{\tilde{V}'(y_o)}(\tilde{V} - \tilde{V}_o) \tag{4.30}$$

is linear to the order of interest, and that $v_o = r_o$ (from (4.29)), so that

$$\tilde{U} \;=\; -4M \ln|\frac{\tilde{V} - \tilde{V}_o}{B}| \;=\; F(\tilde{V}), \tag{4.31}$$

where $B = 3r_o\tilde{V}'(y_o)/av'(y_o)$ is an irrelevant constant. Applying (2.14), it follows that

$$T_{\tilde{U}\tilde{U}}(\tilde{U}) \;\approx\; \frac{1}{192\pi M^2} \tag{4.32}$$

to leading order. The radiation flux is seen to approach a constant as the horizon is approached.

The marginally naked singularity does not exist in this model. The singularity would be marginally naked if the Cauchy horizon coincided with the event horizon for some value of $a$. However, we have seen that the event horizon is given by $y = 2a/3$, which is not a root of the polynomial $f_-(y)$ (except when $a = 0$). The singularity is therefore either naked ($a \leq a_c$) or covered ($a > a_c$), but never marginal.

We now turn to the spectrum of the radiation emitted by the singularities. The famous black body radiation spectrum of the black hole (see Hawking in ref.[4]) is a direct consequence of the form of $F(\tilde{V})$, given in (4.31), and the fact that the integral in (2.9) extends over all of $\mathcal{I}^+$, so that all the outgoing basis states are sampled by the wavepackets formed from incoming plane waves that have scattered through the spacetime.

When a naked singularity is formed, the scattering can occur arbitrarily close to the singularity, so that $F(\tilde{V})$ (as given by (4.19)) has a significantly different

dependence on the advanced coordinate $\tilde{V}$. But there is another and more important difference. If we consider the possibility that a collapsing star does not eliminate all of the spacetime to its future, but that the spacetime continues as the analytic extension of the spacetime in the past of the Cauchy horizon and that $\mathcal{I}^+$ continues to be well defined in its retarded future (and is therefore complete), then since no outgoing wavepackets formed from infalling plane waves are able to reach $\mathcal{I}^+$ in the retarded future of the Cauchy horizon we see that not all of the outgoing basis states on $\mathcal{I}^+$ are sampled by the outgoing wavepackets. A direct consequence of this is that the spectrum is non-thermal. A simple calculation of the Bogoliubov coefficients, with $F(\tilde{V})$ given in (4.19), yields,[8]

$$|\beta(\omega',\omega)|^2 \;=\; \frac{1}{4\pi^2\omega\omega'}|\sum_{k=0}^{\infty}\frac{(i\omega'(B\omega)^{-1/\gamma}e^{i\pi/2\gamma})^k}{k!}\Gamma(\frac{k}{\gamma}+1)|^2, \qquad (4.33)$$

or

$$|\beta(\omega',\omega)|^2 \;=\; \frac{\omega B}{4\pi^2\gamma\omega'^{\gamma+1}}|\sum_{k=0}^{\infty}\frac{(i\omega B\omega'^{-\gamma}e^{-i\pi\gamma/2})^k}{k!}\Gamma(k\gamma+1)|^2. \qquad (4.34)$$

The first expression, eq. (4.33) above, is useful to analyze the high frequency limit $(\omega'(B\omega)^{-1/\gamma} \to 0)$ limit of the spectrum, for in this limit it is sufficient to consider only the first term in the series. Integration over $\omega'$ then yields the familiar logarithmic divergence and the spectrum is seen to fall of as $1/\omega$ in the high frequency region. The second expression, eq. (4.34), serves to analyze its low frequency $(\omega'(B\omega)^{-1/\gamma} \to \infty)$ behavior. Integration over $\omega'$ in this limit shows a power law divergence in the infrared. This divergence is associated with the fact that there are an infinite number of quanta in each mode on $\mathcal{I}^+$. The difference between the divergence in the low and high frequency regimes may be associated with the red-shifting of modes in the proximity of the putative Cauchy horizon. Nevertheless, $|\beta(\omega',\omega)|^2$ is seen to be well behaved as a function of $\omega$, falling as $\omega$ when $\omega \to 0$.

As mentioned in the introduction, it is possible that the Cauchy horizon should be regarded as the natural end point of spacetime, so that the analytical continuation we have considered above is not physically acceptable. In this case, $\mathcal{I}^+$ is not complete and $\tilde{U}$ is no longer a good asymptotic coordinate.

Referring back to figure 1, we see that the transformation to asymptotically flat coordinates must take the form

$$U = -2\kappa e^{-\tilde{U}/2\kappa} = -2\kappa e^{-\tilde{u}/2\kappa} + U^{(0)}, \tag{4.35}$$

which defines the asymptotic null coordinate $\tilde{u}$ and where $U^{(0)}$ is defined by

$$U^{(0)} = -2\kappa e^{-\tilde{U}^{(0)}/2\kappa}, \tag{4.36}$$

in terms of $\tilde{U}^{(0)}$ given earlier. Clearly, its definition is such that $\tilde{u}$ ranges from $-\infty$ to $+\infty$, while $U$ ranges from $-\infty$ to the Cauchy horizon, $U^{(0)}$. A complete outgoing basis set will be defined w.r.t. $\tilde{u}$ instead of $\tilde{U}$. Putting $U = -2\kappa e^{\tilde{U}/2\kappa}$ in (4.35) we find

$$\begin{aligned} \tilde{u} &= \tilde{U}^{(0)} - 2\kappa \ln |1 - e^{-(\tilde{U}-\tilde{U}^{(0)})/2\kappa}| \sim \tilde{U}^{(0)} - 2\kappa \ln |\frac{\tilde{U}-\tilde{U}^{(0)}}{2\kappa}| \\ &\to \tilde{U} - \tilde{U}^{(0)} = 2\kappa e^{-(\tilde{u}-\tilde{U}^{(0)})/2\kappa} \end{aligned} \tag{4.37}$$

(using the fact that we are near the Cauchy horizon). Now relating $\tilde{V}$ and $\tilde{u}$ (instead of $\tilde{U}$ as we did earlier) we find

$$\tilde{u} = F(\tilde{V}) = \tilde{U}_o - \frac{2\kappa}{\gamma} \ln |\frac{\tilde{V}_o - \tilde{V}}{B'}|, \tag{4.38}$$

where $B'$ is an irrelevant constant. However, this is precisely the relationship between the infalling and outgoing coordinates for a black hole, given in (4.31). It

will consequently yield thermal radiation,

$$|\beta(\omega',\omega)|^2 \quad = \quad \frac{\kappa}{\pi\omega'}\frac{1}{e^{4\pi\kappa\omega/\gamma}-1}, \tag{4.39}$$

at the modified temperature, $T \quad = \quad \gamma/4\pi\kappa$. This situation is analogous to the marginally naked singularity treated by Hiscock et. al.[6]

## 5. Discussion

In this paper, we have used the marginally bound, spherically symmetric collapse of inhomogeneous, pressureless dust, which admits both classical black hole and naked singularity end states, to illustrate some key differences between the Hawking radiation from these objects. The central distinguishing feature appears to be the rapid flux of radiation that will be emitted from the naked singularity in the approach to the Cauchy horizon. The intensity of the radiation is clearly a consequence of the large curvatures that are encountered by infalling rays on their way out to future null infinity, so we expect this feature to hold true generically, and whenever regions of high curvature are visible to the asymptotic observer. In this spirit, the naked singularity may be thought of as a region of high curvature that is visible from future null infinity and not necessarily as a true singularity of spacetime. This rapid evaporation signals an instability of the Cauchy horizon, and may cause the collapsing star to evolve in such a way as to avoid its actual formation. It may be the mechanism by which nature avoids naked singularities, in which case the Cosmic Censorship hypothesis would originate in the quantum theory. We have not addressed the manner in which the appearance of the Cauchy horizon may be avoided as this requires a detailed study of the back reaction of spacetime. If nature does in fact employ the quantum theory to avoid naked singularities, the magnified luminosity is likely to be observable and may allow for a glimpse into the behavior of matter fields in strongly curved backgrounds. Again,

if this possibility is taken seriously, it becomes necessary to look for additional features of the radiation that would distinguish naked singularities from other radiating objects. The spectrum of the radiation is one possibility. Although it is to be expected that the electromagnetic spectrum reaching the distant observer will not be characteristic of the collapsing star but rather of the thermalized debris surrounding it, the spectrum radiated in the form of neutrinos and gravitational waves should escape the surrounding matter relatively undisturbed.

A central issue in the calculation of the spectrum of the Hawking radiation from spacetimes that admit Cauchy horizons, is that of the existence of a complete future null infinity. We know of no way to address this question within the semi-classical approach. There are, however, only two logical possibilities: *(i)* either the spacetime continues beyond the Cauchy horizon or *(ii)* it terminates at the Cauchy horizon.

If the local collapse of matter does not destroy the entire universe in its future (it is difficult to imagine that it would), the spacetime can be analytically continued beyond the Cauchy horizon and a complete future null infinity exists. Yet, no incoming waves on past null infinity are able to form wave packets to the future of the Cauchy horizon, therefore only a part of $\mathcal{I}^+$ is actually probed. This results in a non-thermal spectrum. However, it also raises the issue of the consistency of the spectrum derived from the Bogoliubov coefficients. We have addressed this issue by showing explicitly that the total energy radiated, as computed from the integrated spectrum (derived from the Bogoliubov coefficients) is identical to the total energy radiated, as computed from the stress energy tensor to leading order. On the other hand, if the local collapse of a star indeed would destroy the universe in the future of the Cauchy horizon, so that the spacetime must terminate there, we showed that the spectrum of the radiation emitted is then necessarily thermal at a modified temperature.

Only a complete theory of quantum gravity can answer the question of the existence of a complete $\mathcal{I}^+$ as this depends on the final fate of the collapse. We note, however, that the instability of the Cauchy horizon appears to signal that all of $\mathcal{I}^+$ will survive and therefore that the spectrum will be non-thermal.

# References

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